Problem 1  Consider the function $y = f(x)$ given by

$$y = \frac{x^{53} \sqrt[4]{x + 6}}{(x^2 - 2)^{10}}$$

a) Find the derivative of $f(x)$ using logarithmic differentiation. Express it in the form $f'(x) = y \cdot \ldots$

b) Find the exact value of that derivative at $x = 1$.

Solution  a) We take logarithms on both sides:

$$\ln y = 5 \ln x + 4x \ln 3 + \frac{1}{2} \ln(x + 6) - 10 \ln(x^2 - 2).$$

Then we differentiate both sides. Note that $\ln 3$ is a constant, so $4x \ln 3$ is a constant multiple of $x$. And for $\ln(x^2 - 2)$, we need to use the Chain Rule with $u(x) = x^2 - 2, u'(x) = 2x$. So

$$\frac{y'}{y} = \frac{5}{x} + 4 \ln 3 + \frac{1}{2(x + 6)} - \frac{20x}{x^2 - 2}.$$

Multiply both sides by $y$ to get

$$y' = y \left(\frac{5}{x} + 4 \ln 3 + \frac{1}{2(x + 6)} - \frac{20x}{x^2 - 2}\right).$$

b) Substitute $x = 1$ to get $y(1) = 3^4 \sqrt[4]{7} = 81 \sqrt[4]{7}$. And from part a),

$$y'(1) = 81 \sqrt[4]{7} \left(\frac{5 + 4 \ln 3 + \frac{1}{14} + 20}{25 \frac{1}{14} + 4 \ln 3}\right) = 81 \sqrt[4]{7} \left(\frac{351}{14} + 4 \ln 3\right)$$