Math 165 - Practice Exam 2 - solutions

Problem 1 Find the derivatives of the following functions.

\[
\begin{align*}
    u &= 3x^6 + 5x^4 - 5x + 2000 \\
    y &= \frac{3x^2 + 2x + 1}{2x^4 - 1} \\
    z &= \tan x + \sin x \\
    w &= \frac{x \sin x - 3}{x^2 + 1}
\end{align*}
\]

Use only Rules for Differentiation and derivatives of basic functions. No simplifying necessary.

Solution.

\[
\begin{align*}
    \frac{du}{dx} &= 18x^5 + 20x^3 - 5 \\
    \frac{dy}{dx} &= \frac{(2x^4 - 1)(6x + 2) - 8x^3(3x^2 + 2x + 1)}{(2x^4 - 1)^2} \\
    \frac{dz}{dx} &= \sec^2 x + \cos x. \\
    \frac{dw}{dx} &= \frac{(\sin x + x \cos x)(x^2 + 1) - 2x(x \sin x - 3)}{(x^2 + 1)^2}.
\end{align*}
\]

Problem 2 A point \( P = (x, y) \) is traveling on the perimeter of a wheel with radius 14 inches. The origin \( O \) of the coordinate system is at the center of the wheel.

a) Express the angle \( \theta \) which the ray \( OP \) makes with the positive \( x \)-axis as a function of the \( x \)-coordinate of \( P \), given that the angle is in \( [0, \pi] \). What is the angle when \( x = -7 \) inches?

b) Find the derivative of \( \theta \) with regard to \( x \) when \( x = -7 \) inches.

Solution. a) Since \( x = 14 \cos \theta \),
we can solve this for $\theta$ in $[0, \pi]$ using the inverse cosine,

$$\theta = \arccos\left(\frac{x}{14}\right).$$

When $x = -7$,

$$\theta = \arccos(-0.5) = \frac{2\pi}{3}.$$

b) 

$$\frac{d\theta}{dx} = -\frac{1}{14\sqrt{1-(x/14)^2}} = -\frac{1}{\sqrt{14^2-x^2}}.$$

When $x = -7$, this equals $-1/(7\sqrt{3}) = -\sqrt{3}/21$.

**Problem 3** Suppose a function $y = f(x)$ is such that

$$y - \frac{\pi}{2} + 2 \sin y = x.$$

a) Find the derivative of $f(x)$ at $x = 2$ given that $f(2) = \pi/2$.
b) Find an equation for the tangent line to the graph of $f(x)$ at $x = 2$.

**Solution.** Note that we have no way of getting an explicit formula for $f(x)$.

a) We differentiate both sides using implicit differentiation:

$$y' - 2y' \cos y = 1.$$

We can solve this for $y'$:

$$y' = \frac{1}{1 - 2 \cos y}$$

and then substitute $y = \pi/2$ to get $y' = f'(2) = 1$ at the point when $x = 2$ and $y = \pi/2$.
b) We just computed the slope of that tangent line, it is 1! So it has equation

$$y = 1 \cdot (x - 2) + \pi/2.$$

**Problem 4** A bike wheel of diameter 28 inches is spinning counterclockwise in one place at 10 revolutions per minute. Place the origin of a coordinate system at the center of the wheel and let $P$ be the point $(20, 0)$ (with unit 1 inch on both axes). How fast is the distance between $P$ and a point on the
perimeter of the wheel changing at the time when the point is at its highest position? Give exact answers.

Solution. Name the relevant quantities: Say $Q$ is the point on the perimeter of the wheel, with coordinates $x, y$ that are functions of time $t$. The distance between $P$ and $Q$ is the length of the hypotenuse of a right triangle, so

$$D^2 = (20 - x)^2 + y^2.$$  

Also, $Q$ sits on the perimeter of a circle of radius 14, so

$$x^2 + y^2 = 14^2$$

which means we can simplify the equation for $D$ to

$$D^2 = 20^2 - 40x + 14^2.$$  

Now, differentiate both sides with regard to time $t$:

$$2DD' = -40x'.$$  

We'll still have to figure out $D$ and $x'$ at the given time. First, when $Q$ is at the highest position, we have $x = 0$ and $y = 14$, so

$$D = \sqrt{20^2 + 14^2} = \sqrt{596}.$$  

If we write $\theta$ for the angle that $OQ$ makes with the positive $x$-axis, then

$$x = 14 \cos \theta$$

and therefore

$$x' = -14 \sin(\theta) \theta'.$$

like in the last problem. Next, the clue '10 counterclockwise revolutions per minute' means that the angle $\theta$ is changing at a constant, positive rate,

$$\theta' = 10 \cdot 2\pi$$

(in radians per minute). So,

$$x' = -14 \sin(\theta) \cdot 10 \cdot 2\pi.$$
At the highest position for $Q$, $\theta = \pi/2$, so the sine equals 1. Put all the pieces together to get

$$D' = -\frac{20x'}{D} = \frac{20 \cdot 14 \cdot 10 \cdot 2\pi}{\sqrt{596}} \approx 720.63$$

in inches per minute.

**Problem 5** Consider the function

$$y = \sqrt[10]{\frac{3x + 4}{2x - 4}}.$$  

a) Simplify $\ln y$, using rules for logarithms, as much as possible.

b) Find $y'$ using logarithmic differentiation.

**Solution.** For a),

$$\ln y = \frac{1}{10} (\ln(3x + 4) - \ln(2x - 4)).$$

For b), differentiate both sides above:

$$\frac{y'}{y} = \frac{1}{10} \left( \frac{3}{3x + 4} - \frac{2}{2x - 4} \right).$$

Then multiply both sides by $y$ (and STOP RIGHT THERE).